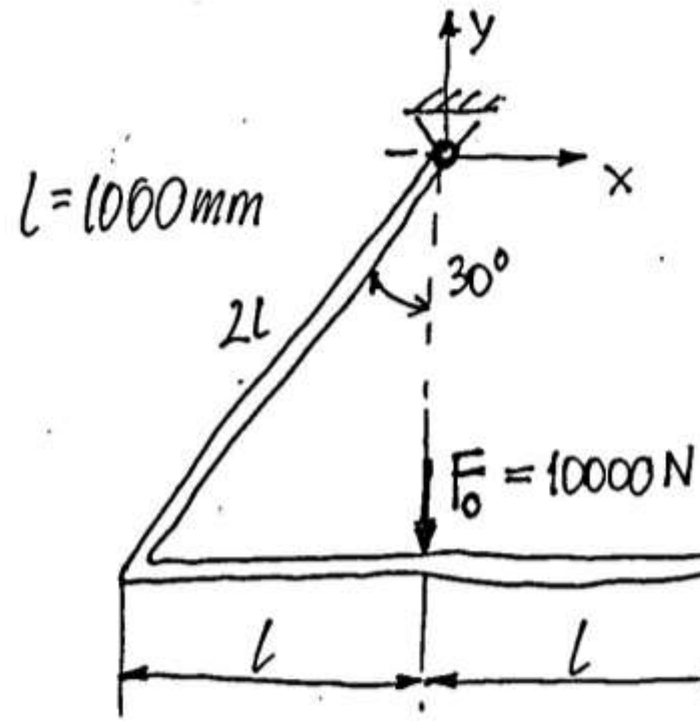
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

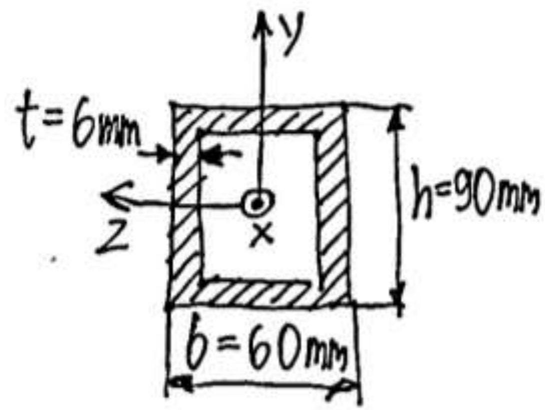
Finite element method (FEM1)

Lecture 10C. Frames - examples

05.2025

Example: Build a 2D FEM model of the frame. Determine nodal displacements, stresses, internal forces and reactions. Check equilibrium conditions.





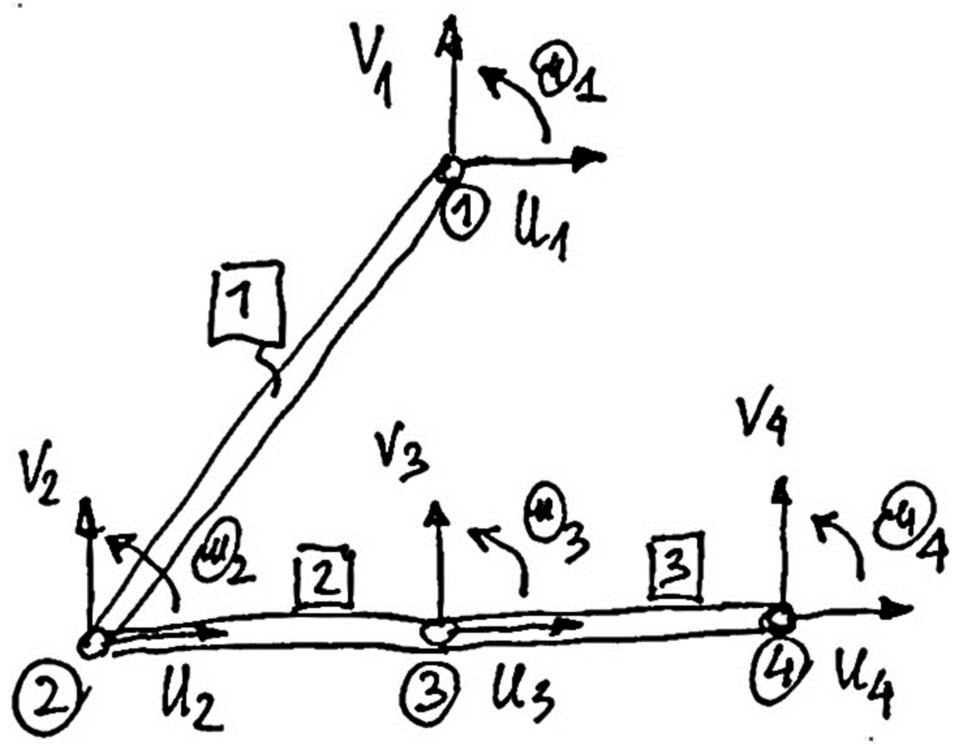
$$A = bh - (b-2t)(h-2t) = 1656 \text{ mm}^2$$

$$J_z = \frac{bh^3}{12} - \frac{(b-2t)(h-2t)^3}{12} = 1.7468 \cdot 10^6 \text{ mm}^4$$

Nodal parameters:

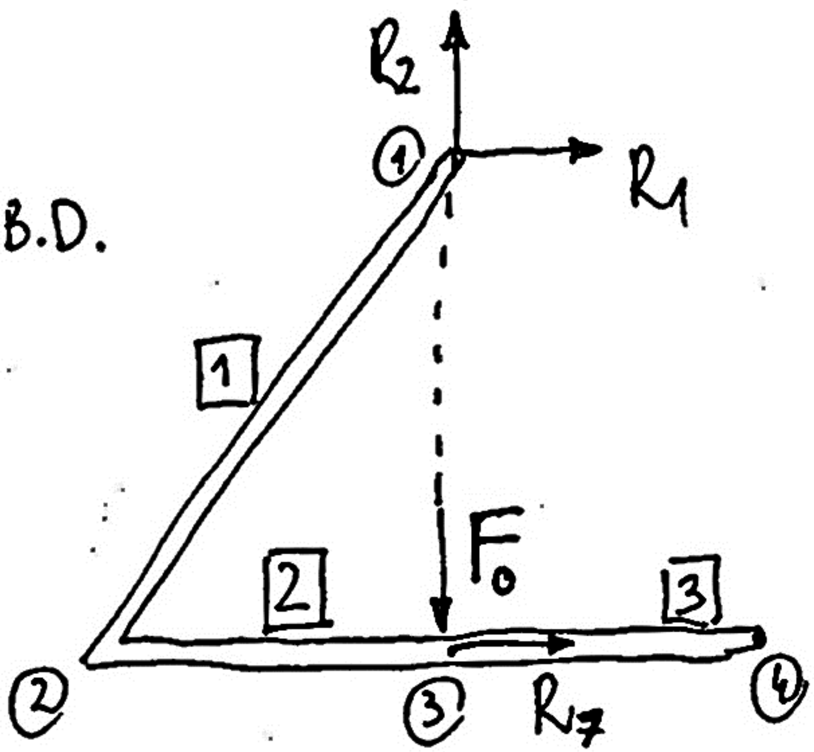
$$\underset{12\times1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

$$u_1 = 0$$
$$v_1 = 0$$
$$u_3 = 0$$



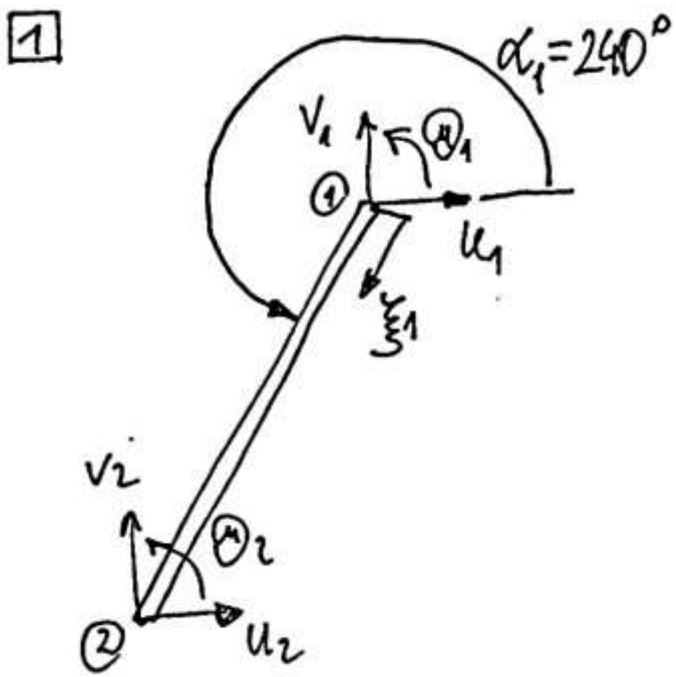
Loads and reactions:

F.B.D.



$$\underset{12\times 1}{\{F\}} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ -F_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stiffness matrices of element 1:



$$C_1 = -\frac{1}{2}$$

$$S_1 = -\frac{\sqrt{3}}{2}$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, \vartheta_1, u_2, v_2, \vartheta_2 \rfloor$$

$$\underset{6\times 6}{[k]_1} = \begin{bmatrix} \frac{EA}{2L} & 0 & 0 & -\frac{EA}{2L} & 0 & 0 \\ 0 & \frac{12EJ_z}{8L^3} & \frac{6EJ_z}{4L^2} & 0 & -\frac{12EJ_z}{8L^3} & \frac{6EJ_z}{4L^2} \\ 0 & \frac{6EJ_z}{4L^2} & \frac{4EJ_z}{2L} & 0 & -\frac{6EJ_z}{4L^2} & \frac{2EJ_z}{2L} \\ -\frac{EA}{2L} & 0 & 0 & \frac{EA}{2L} & 0 & 0 \\ 0 & -\frac{12EJ_z}{8L^3} & -\frac{6EJ_z}{4L^2} & 0 & \frac{12EJ_z}{8L^3} & -\frac{6EJ_z}{4L^2} \\ 0 & \frac{6EJ_z}{4L^2} & \frac{2EJ_z}{2L} & 0 & -\frac{6EJ_z}{4L^2} & \frac{4EJ_z}{2L} \end{bmatrix}$$

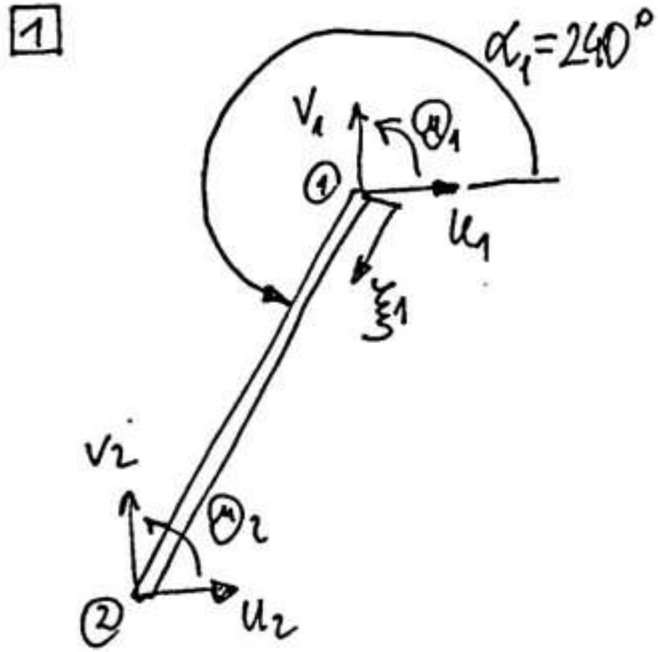
Transformation matrix of element 1:

$$\underset{6\times 6}{[T_f]_e} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{[T_f]_1} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{[T_f]_1^T} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

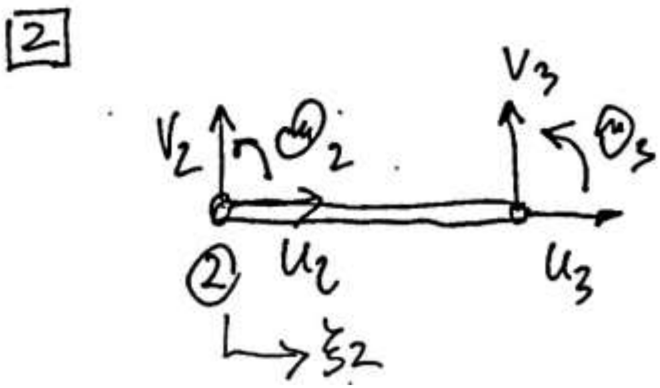
Extended transformation matrix of element 1:



$$\underset{6\times 6}{[k_g]_1} = \underset{6\times 6}{[T_f]_1^T}\,\underset{6\times 6}{[k]_1}\,\underset{6\times 6}{[T_f]_1}$$

$$\underset{12\times 12}{[k_g]_1^*} = \left[\begin{array}{c|c} \underset{6\times 6}{[k_g]_1} & \underset{6\times 6}{[0]} \\ \hline \underset{6\times 6}{[0]} & \underset{6\times 6}{[0]} \end{array}\right]$$

Stiffness matrices of element 2:



$$\alpha_2 = 0^\circ, \quad c_2 = 1, \quad s_2 = 0$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_2} = \lfloor u_2, v_2, \Theta_2, u_3, v_3, \Theta_3 \rfloor$$

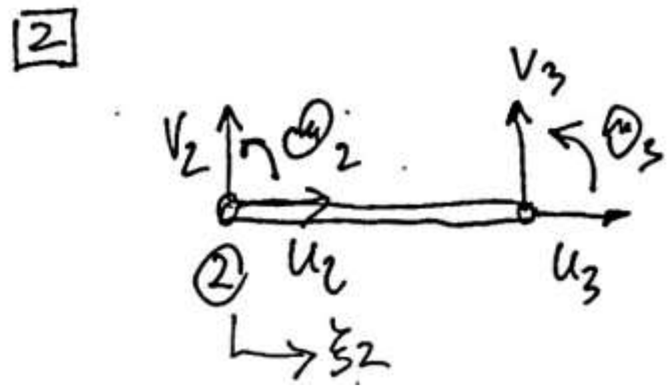
$$\underset{6\times 6}{[k]_2} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EJ_z}{l^3} & \frac{6EJ_z}{l^2} & 0 & -\frac{12EJ_z}{l^3} & \frac{6EJ_z}{l^2} \\ 0 & \frac{6EJ_z}{l^2} & \frac{4EJ_z}{l} & 0 & -\frac{6EJ_z}{l^2} & \frac{2EJ_z}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EJ_z}{l^3} & -\frac{6EJ_z}{l^2} & 0 & \frac{12EJ_z}{l^3} & -\frac{6EJ_z}{l^2} \\ 0 & \frac{6EJ_z}{l^2} & \frac{2EJ_z}{l} & 0 & -\frac{6EJ_z}{l^2} & \frac{4EJ_z}{l} \end{bmatrix}$$

Transformation matrix of element 2:

$$\underset{6\times 6}{[T_f]_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \underset{6\times 6}{[T_f]_2^T}$$

$$\underset{6\times 6}{[k_g]_2} = \underset{6\times 6}{[T_f]_2^T}\,\underset{6\times 6}{[k]_2}\,\underset{6\times 6}{[T_f]_2} = \underset{6\times 6}{[k]_2}$$

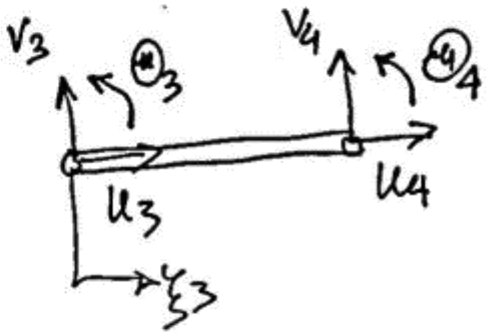
Extended transformation matrix of element 2:



$$\underset{6\times 6}{[K_g]_2^{*}} = \left[\begin{array}{c:c:c} \underset{3\times 3}{[0]} & \underset{3\times 6}{[0]} & \underset{3\times 3}{[0]} \\ \hdashline \underset{6\times 3}{[0]} & \underset{6\times 6}{[K_g]_2} & \underset{6\times 3}{[0]} \\ \hdashline \underset{3\times 3}{[0]} & \underset{3\times 6}{[0]} & \underset{3\times 3}{[0]} \end{array}\right]$$

Extended transformation matrix of element 3:

$\boxed{3}$



$$\alpha_3 = 0^\circ \;,\; C_3 = 1,\; S_3 = 0$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_3} = \lfloor u_3, v_3, \theta_3, u_4, v_4, \theta_4 \rfloor$$

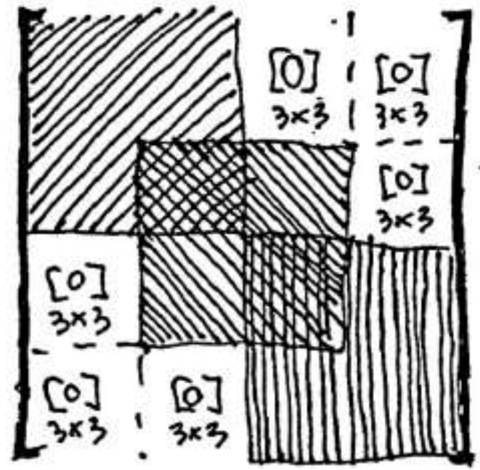
$$\underset{6\times 6}{[K]_3} = \underset{6\times 6}{[k]_2} \quad , \quad \underset{6\times 6}{[T_f]_3} = \underset{6\times 6}{[T_f]_3^T} = \underset{6\times 6}{[T_f]_2}$$

$$\underset{6\times 6}{[k_g]_3} = \underset{6\times 6}{[k_g]_2} = \underset{6\times 6}{[k]_3}$$

$$\underset{6\times 6}{[k_g]_3^*} = \begin{bmatrix} \underset{6\times 6}{[0]} & \underset{6\times 6}{[0]} \\ \underset{6\times 6}{[0]} & \underset{6\times 6}{[k_g]_3} \end{bmatrix}$$

Global Stiffness Matrix:

$$\underset{12\times 12}{[K]} = \underset{12\times 12}{[K_g]^*_1} + \underset{12\times 12}{[K_g]^*_2} + \underset{12\times 12}{[K_g]^*_3} =$$



System of equations:

$$\underset{12\times 12}{[K]} \cdot \underset{12\times 1}{\{q\}} = \underset{12\times 1}{\{F\}}$$

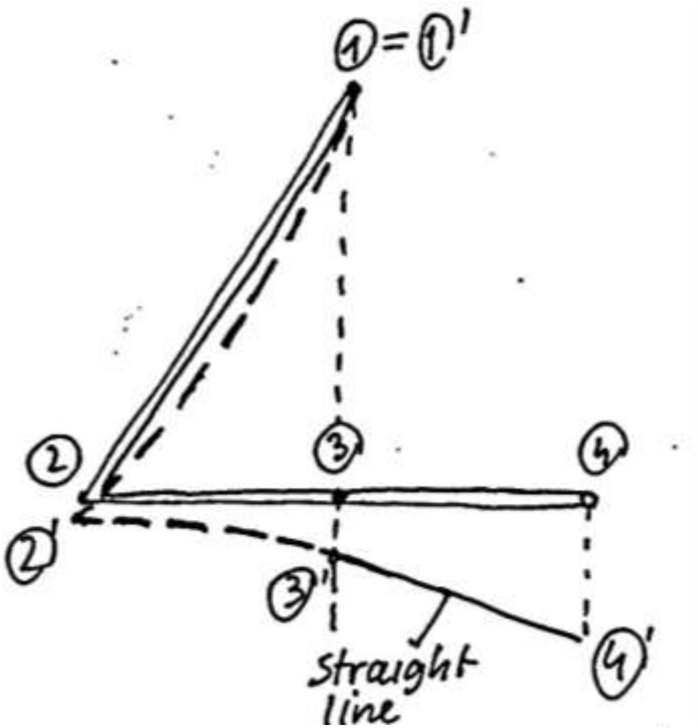
+ boundary conditions

$u_1 = 0, \; v_1 = 0, \; u_3 = 0$

$$\underset{9\times 9}{[K]} \cdot \begin{Bmatrix} \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \\ u_4 \\ v_4 \\ \varphi_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

($\underset{9\times 9}{[K]} = \underset{12\times 12}{[K]}$ without rows and columns: no. 1, 2, 7)

Nodal displacements:



$$\underset{9\times 1}{\{q\}} = \underset{9\times 9}{[K]^{-1}} \cdot \underset{9\times 1}{\{F\}} = \left\{ \begin{array}{r} 0.00956 \\ 0 \\ -0.0604 \\ -0.01907 \\ -28.6692 \\ -0.0334 \\ 0 \\ -62.0486 \\ -0.0334 \end{array} \right\} \begin{array}{l} (rad) \\ (mm) \\ (mm) \\ (rad) \\ (mm) \\ (rad) \\ (mm) \\ (mm) \\ (rad) \end{array} = \left\{ \begin{array}{c} \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{array} \right\}$$

Reactions:

$$\underset{12\times 12}{\left[K\right]}\cdot\underset{12\times 1}{\left\{q\right\}} = \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ -F_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

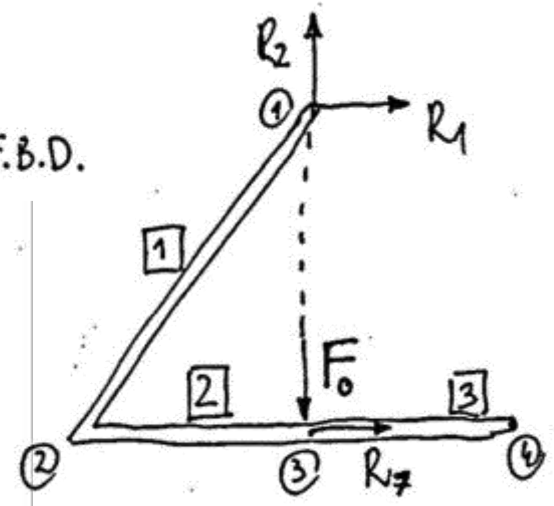
$\Rightarrow R_1 = 0\ \text{N}$

$\Rightarrow R_2 = 10000\ \text{N}$

$\Rightarrow R_7 = 0\ \text{N}$

(additional constraint $u_3 = 0$ did not change the deformation)

F.B.D.



Displacements of element 1 in the local coordinate system:

$\boxed{1}$

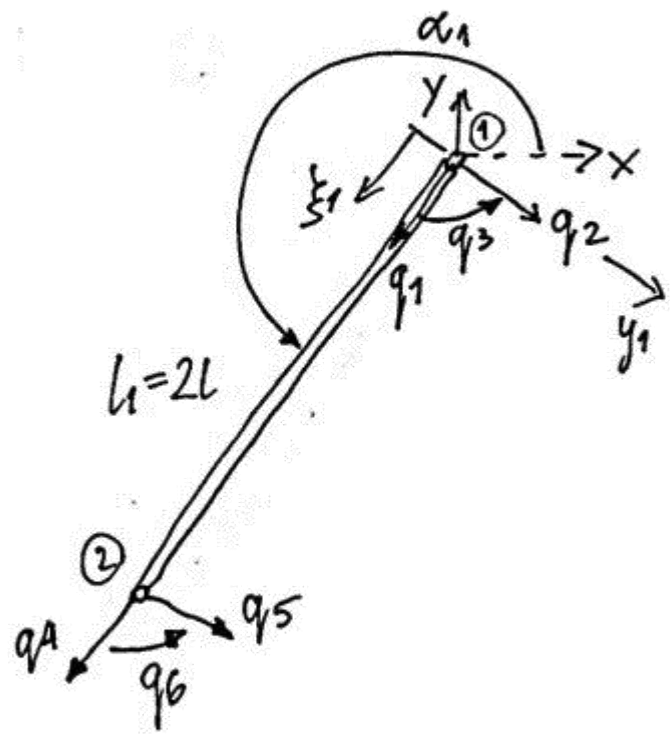
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \underset{6\times 6}{\left[T_f\right]_1} \cdot \underset{6\times 1}{\{q_g\}_1} = \underset{6\times 6}{\left[T_f\right]_1} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}_1 =$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0.00956 \\ 0 \\ -0.0604 \\ -0.01907 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0.00956 \\ 0.0523 \\ 0.0302 \\ -0.01907 \end{Bmatrix}_1$$

Internal stresses and forces in element 1:

$\boxed{1}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0.00956 \\ 0.0523 \\ 0.0302 \\ -0.01907 \end{Bmatrix}_1$$



Tension in elment 1:

$$\sigma_1 = \frac{E}{2l}\left(q_4 - q_1\right)_1 = 5.23\,MPa$$

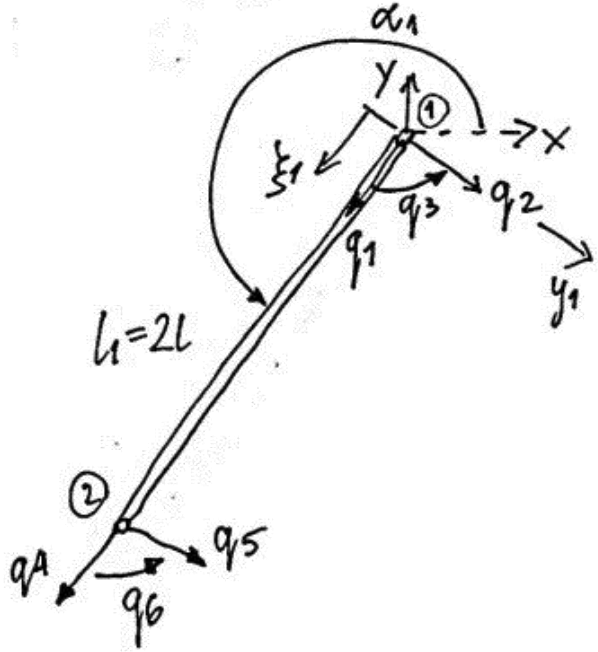
$$N_1 = \sigma_1 \cdot A = 8660.254\ N$$

Internal forces in element 1 due to bending:

$$M_{z_1}(\xi_1) = EJ_z \cdot w_1'' = EJ_z \cdot \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_1 =$$

$$= EJ_z \left\lfloor -\frac{6}{l_1^2} + \frac{12}{l_1^3}\xi_1 ,\ -\frac{4}{l_1} + \frac{6}{l_1^2}\xi_1 ,\ \frac{6}{l_1^2} - \frac{12}{l_1^3}\xi_1 ,\ -\frac{2}{l_1} + \frac{6}{l_1^2}\xi_1 \right\rfloor \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_5 \\ q_6 \end{Bmatrix}_1 \Rightarrow$$

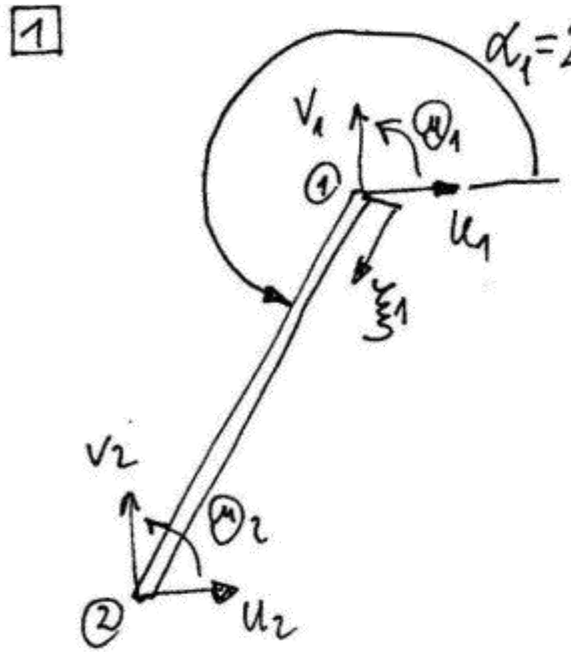
$$\Rightarrow \quad M_{z_1}(0) = 0 \quad , \quad M_{z_1}(2l) = -1\cdot 10^7 \text{ Nmm}$$

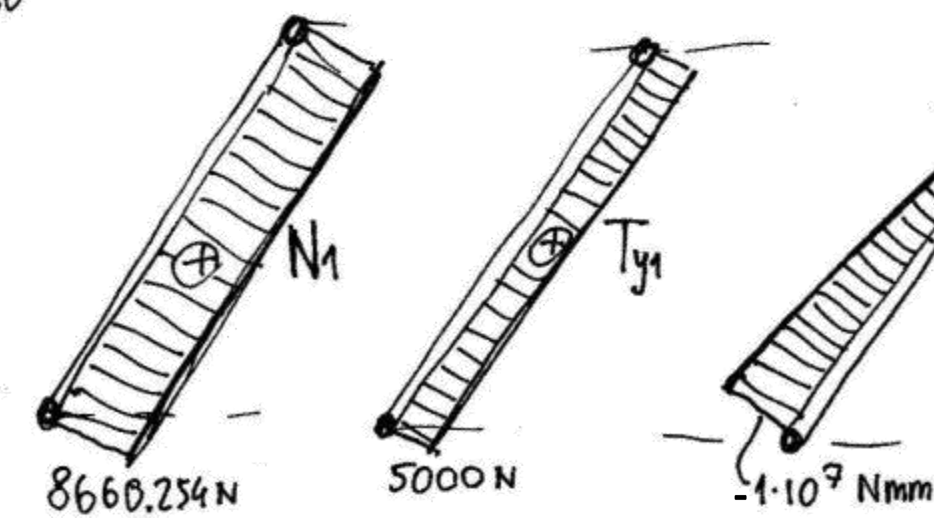


$$T_{y_1} = -EJ_z \cdot w_1''' = -EJ_z \cdot \underset{1\times 4}{\lfloor N''' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_1 =$$

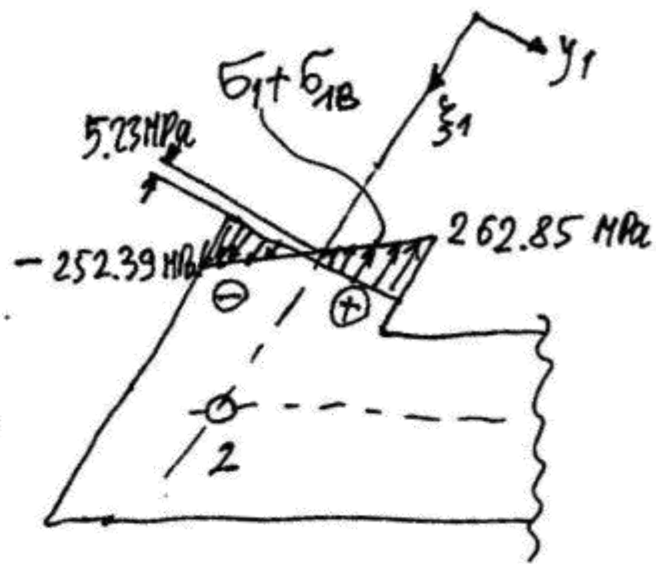
$$= -EJ_z \left\lfloor \frac{12}{l_1^3} , \frac{6}{l_1^2} , -\frac{12}{l_1^3} , \frac{6}{l_1^2} \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_1 = 5000\,N$$

Internal forces and stresses in element 1 (resultant)





$$\sigma_{1B}(\xi_1, y_1) = -\frac{M_{z1}(\xi_1)\cdot y_1}{J_z}$$



$$\sigma_{1B}\left(2l, \frac{h}{2}\right) = \frac{-(-1\cdot 10^7 \, Nmm)\cdot \frac{h}{2}}{J_z} = 257{,}62 \, MPa$$

Internal stresses and forces in element 2:

$\boxed{2}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \underset{6\times 6}{[T_f]_2} \cdot \underset{6\times 1}{\{q_g\}_2} = \underset{6\times 6}{[T_f]_2} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ \Theta_2 \\ u_3 \\ v_3 \\ \Theta_3 \end{Bmatrix}_2 =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ -0.0604 \\ -0.01907 \\ 0 \\ -28.6692 \\ -0.03334 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ -0.0604 \\ -0.01907 \\ 0 \\ -28.6692 \\ -0.03334 \end{Bmatrix}_2$$

Internal stresses and forces in element 2:

$\boxed{2}$

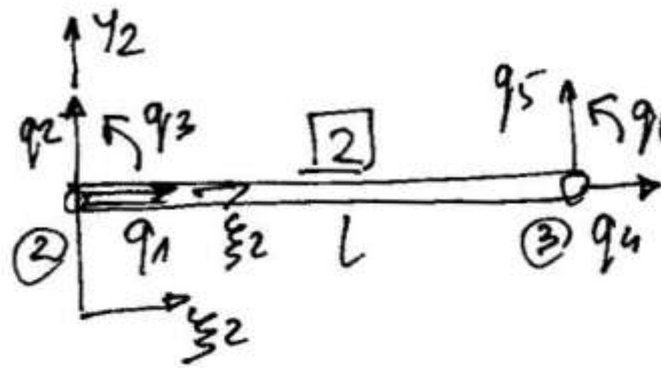
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ -0.0604 \\ -0.01907 \\ 0 \\ -28.6692 \\ -0.03334 \end{Bmatrix}_2$$

Tension in element 2:

DAK.

$$\sigma_2 = \frac{E}{L}\left(q_4 - q_1\right)_2 = 0$$

$$N_2 = \sigma_2 \cdot A = 0$$



Internal forces in element 2 due to bending:

$$M_{z_2}(\xi_2) = EJ_2 \cdot w_2'' = EJ_2 \cdot \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 =$$

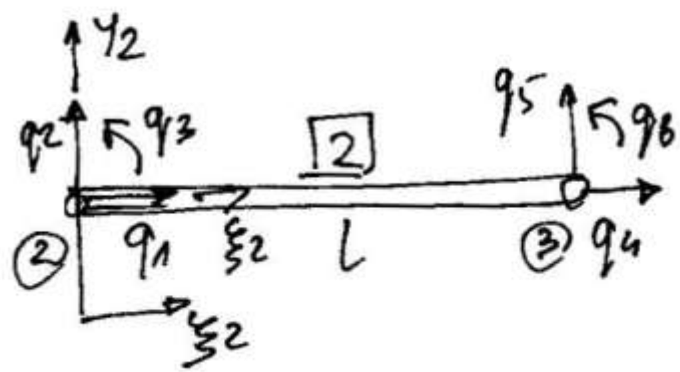
$$= EJ_2 \left\lfloor -\frac{6}{l^2} + \frac{12}{l^3}\xi_2, -\frac{4}{l} + \frac{6}{l^2}\xi_2, \frac{6}{l^2} - \frac{12}{l^3}\xi_2, -\frac{2}{l} + \frac{6}{l^2}\xi_2 \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 \Rightarrow$$

$$M_{z_2}(0) = -1 \cdot 10^7 \, \text{Nmm} \quad ; \quad M_{z_2}(l) = 0$$

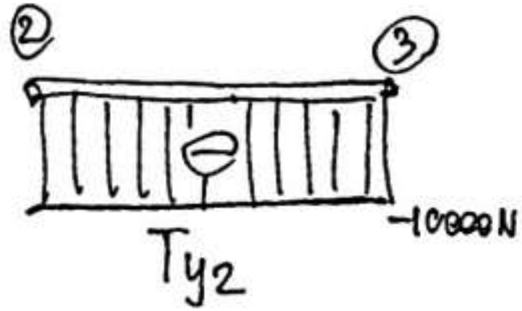
$$T_{y_2} = -EJ_2 \cdot w_2''' = -EJ_2 \lfloor N''' \rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 =$$

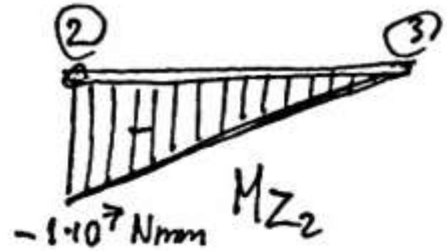
$$= -EJ_2 \left\lfloor \frac{12}{l^3}, \frac{6}{l^2}, -\frac{12}{l^3}, \frac{6}{l^2} \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_2 =$$

$$= -10000 \, \text{N}$$



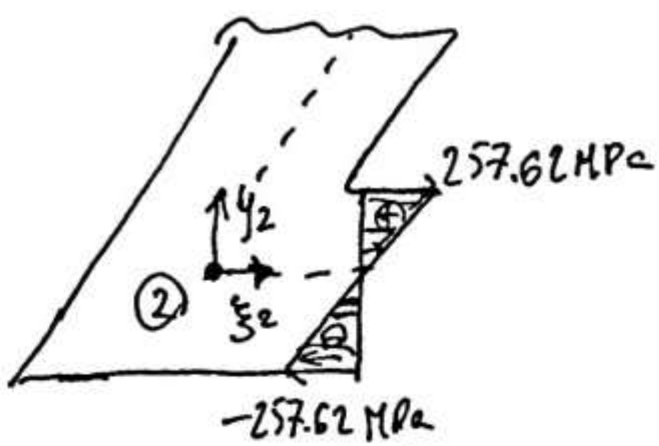
Internal forces and stresses in element 2 (resultant)





$$\sigma_{2B}(\xi_2, y_2) = -\frac{M_{Z_2}(\xi_2) \cdot y_2}{J_z}$$

$$\sigma_{2B}\left(0, \frac{h}{2}\right) = -\frac{(-1 \cdot 10^7\,Nmm) \cdot \frac{h}{2}}{J_z} = 257.62\,MPa$$



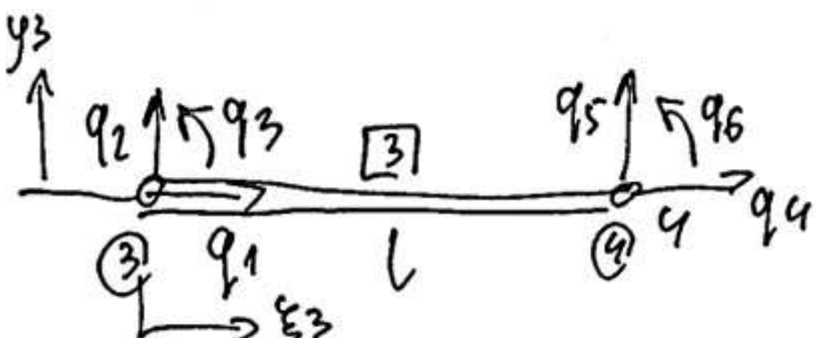
Internal stresses and forces in element 3:

$\boxed{3}$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_3 = \underset{6\times 6}{[T_f]_3} \cdot \underset{6\times 1}{\{q_g\}_3} = \underset{6\times 6}{[1]} \cdot \begin{Bmatrix} u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ -28.6692 \\ -0.0334 \\ 0 \\ -62.0486 \\ -0.0334 \end{Bmatrix}_3$$

Tension in element 3:

$$\sigma_3 = \frac{E}{l}(q_4 - q_1)_3 = 0$$



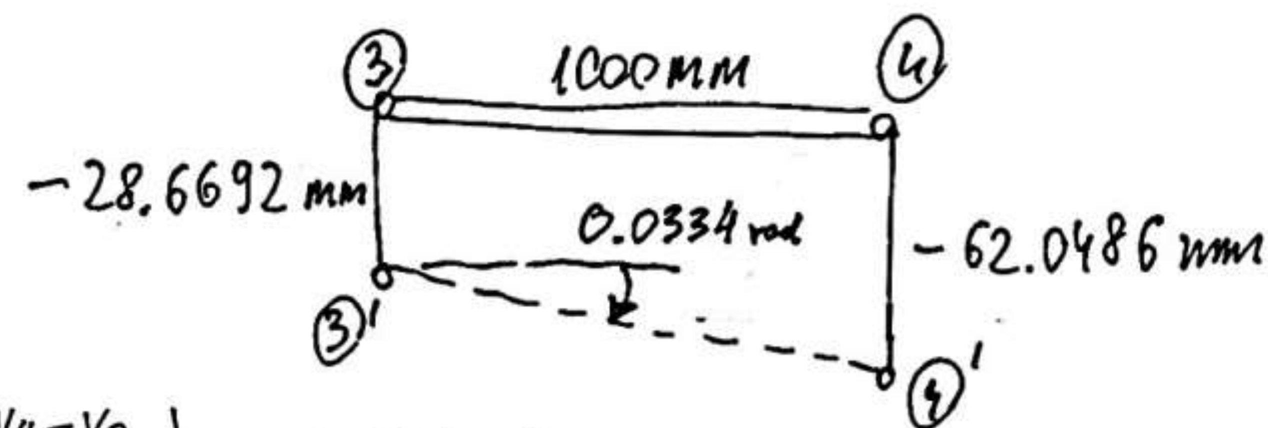
$$N_3 = \sigma_3 \cdot A = 0$$

Internal forces in element 3 due to bending:

$$M_{z3}(\xi_3) = EJ_z \cdot w_3'' = EJ_z \cdot \underset{1\times 4}{\lfloor N'' \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_3 =$$

$$= EJ_z \left\lfloor -\frac{6}{l^2} + \frac{12}{l^3}\xi_3,\ -\frac{4}{l} + \frac{6}{l^2}\xi_3,\ \frac{6}{l^2} - \frac{12}{l^3}\xi_3,\ -\frac{2}{l} + \frac{6}{l^2}\xi_3 \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_3 \\ q_5 \\ q_6 \end{Bmatrix}_3 =$$

$$= 0 \Rightarrow \sigma_{3B}(\xi_3, y_3) = 0$$



$$\operatorname{atan}\left(\frac{v_4 - v_3}{l}\right) = -0.0334 \text{ rad}$$

Equilibrium: